Introduction to Algorithmic Trading Strategies
Lecture 4

Optimal Pairs Trading by Stochastic Control

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Outline

- Problem formulation
- Ito's lemma
- Dynamic programming
- Hamilton-Jacobi-Bellman equation
- Riccati equation
- Integrating factor
Reference

Basket Creation vs. Trading

- In lecture 3, we discussed a few ways to construct a mean-reverting basket.
- In this and the next lectures, we discuss how to trade a mean-reverting asset, if such exists.
Stochastic Control

- We model the difference between the log-returns of two assets as an Ornstein-Uhlenbeck process.
- We compute the optimal position to take as a function of the deviation from the equilibrium.
- This is done by solving the corresponding Hamilton-Jacobi-Bellman equation.
Assume a risk free asset $M_t$, which satisfies

$$dM_t = rM_t dt$$

Assume two assets, $A_t$ and $B_t$.

Assume $B_t$ follows a geometric Brownian motion.

$$dB_t = \mu B_t dt + \sigma B_t dz_t$$

$x_t$ is the spread between the two assets.

$$x_t = \log A_t - \log B_t$$
Ornstein-Uhlenbeck Process

- We assume the spread, the basket that we want to trade, follows a mean-reverting process.
  \[ dx_t = k(\theta - x_t)dt + \eta d\omega_t \]
- \( \theta \) is the long term equilibrium to which the spread reverts.
- \( k \) is the rate of reversion. It must be positive to ensure stability around the equilibrium value.
Let $\rho$ denote the instantaneous correlation coefficient between $z$ and $\omega$.

$$E[d\omega_t dz_t] = \rho dt$$
Univariate Ito’s Lemma

- Assume
  - \( dX_t = \mu_t \, dt + \sigma_t \, dB_t \)
  - \( f(t, X_t) \) is twice differentiable of two real variables
- We have
  - \[ df(t, X_t) = \left( \frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2} \right) \, dt + \sigma_t \frac{\partial f}{\partial x} \, dB_t \]
Log example

- For G.B.M., $dX_t = \mu X_t dt + \sigma X_t dz_t$, $d \log X_t =$?
- $f(x) = \log(x)$
- $\frac{\partial f}{\partial t} = 0$
- $\frac{\partial f}{\partial x} = \frac{1}{x}$
- $\frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$

- $d \log X_t = \left(\mu X_t \frac{1}{X_t} + \frac{(\sigma X_t)^2}{2} \left(-\frac{1}{X_t^2}\right)\right) dt + \sigma X_t \left(\frac{1}{X_t}\right) dB_t$
- $= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB_t$
Multivariate Ito’s Lemma

- Assume
  - \( X_t = (X_{1t}, X_{2t}, \cdots, X_{nt}) \) is a vector Ito process
  - \( f(x_{1t}, x_{2t}, \cdots, x_{nt}) \) is twice differentiable

- We have
  \[
  df(X_{1t}, X_{2t}, \cdots, X_{nt}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(X_{1t}, X_{2t}, \cdots, X_{nt}) dX_i(t)
  \]
  \[
  + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(X_{1t}, X_{2t}, \cdots, X_{nt}) d[X_i, X_j](t)
  \]
Multivariate Example

- \( \log A_t = x_t + \log B_t \)
- \( A_t = \exp(x_t + \log B_t) \)
- \( \frac{\partial A_t}{\partial x_t} = \exp(x_t + \log B_t) = A_t \)
- \( \frac{\partial A_t}{\partial B_t} = \exp(x_t + \log B_t) \frac{1}{B_t} = \frac{A_t}{B_t} \)
- \( \frac{\partial^2 A_t}{\partial x_t^2} = \frac{\partial A_t}{\partial x_t} = A_t \)
- \( \frac{\partial^2 A_t}{\partial B_t^2} = \frac{\partial}{\partial B_t} \left( \frac{A_t}{B_t} \right) = 0 \)
- \( \frac{\partial^2 A_t}{\partial B_t \partial x_t} = \frac{\partial}{\partial B_t} \left( \frac{\partial A_t}{\partial x_t} \right) = \frac{\partial A_t}{\partial B_t} = \frac{A_t}{B_t} \)
What is the Dynamic of Asset $A_t$?

\[
\partial A_t = \frac{\partial A_t}{\partial x} dx_t + \frac{\partial A_t}{\partial B_t} dB_t + \frac{1}{2} \frac{\partial^2 A_t}{\partial x^2} (dx_t)^2 + \frac{\partial^2 A_t}{\partial B_t \partial x} (dx_t)(dB_t)
\]

\[
= A_t dx_t + \frac{A_t}{B_t} dB_t + \frac{1}{2} A_t (dx_t)^2 + \frac{A_t}{B_t} (dx_t)(dB_t)
\]

\[
= A_t[k(\theta - x_t)dt + \eta d\omega_t] + \frac{A_t}{B_t} [\mu B_t dt + \sigma B_t dz_t] + \frac{1}{2} A_t \eta^2 dt + \frac{A_t}{B_t} \rho \eta \sigma B_t dt
\]
Dynamic of Asset $A_t$

\[ \partial A_t = A_t[k(\theta - x_t)dt + \eta d\omega_t] + A_t[\mu dt + \sigma dz_t] + \frac{1}{2} A_t \eta^2 dt + A_t \rho \eta \sigma dt \]

\[ = A_t \left[ k(\theta - x_t) + \mu + \frac{1}{2} \eta^2 + \rho \eta \sigma \right] dt + A_t \eta d\omega_t + A_t \sigma dz_t \]

\[ = A_t \left\{ \left[ \mu + k(\theta - x_t) + \frac{1}{2} \eta^2 + \rho \eta \sigma \right] dt + \sigma dz_t + \eta d\omega_t \right\} \]
Notations

- \( V_t \): the value of a self-financing pairs trading portfolio
- \( h_t \): the portfolio weight for stock A
- \( \tilde{h}_t = -h_t \): the portfolio weight for stock B
Self-Financing Portfolio Dynamic

\[
\frac{dV_t}{V_t} = h_t \frac{dA_t}{A_t} +  \tilde{h_t} \frac{dB_t}{B_t} + \frac{dM_t}{M_t} \\
= h_t \left\{ \left[ \mu + k(\theta - x_t) + \frac{1}{2} \eta^2 + \rho \eta \sigma \right] dt + \sigma dZ_t + \eta d\omega_t \right\} - h_t \{ \mu dt + \sigma dZ_t \} + r dt \\
= h_t \left\{ \left[ k(\theta - x_t) + \frac{1}{2} \eta^2 + \rho \eta \sigma \right] dt + \eta d\omega_t \right\} + r dt \\
= \left\{ h_t \left[ k(\theta - x_t) + \frac{1}{2} \eta^2 + \rho \eta \sigma \right] + r \right\} dt + h_t \eta d\omega_t
Power Utility

- **Investor preference:**
  
  \[ U(x) = x^\gamma \]
  
  \[ x \geq 0 \]
  
  \[ 0 < \gamma < 1 \]
Problem Formulation

- \( \max \limits_{h_t} E[V_T^r] \), s.t.,
- \( V(0) = v_0, x(0) = x_0 \)
- \( dx_t = k(\theta - x_t)dt + \eta d\omega_t \)
- \( dV_t = h_t dx_t = h_t k(\theta - x_t)dt + h_t \eta d\omega_t \)
- Note that we simplify GBM to BM of \( V_t \), and remove some constants.
Consider a stage problem to minimize (or maximize) the accumulated costs over a system path.

Cost = $c_3 + \sum_{t=0}^{2} c_t$
Dynamic Programming Formulation

- State change: $x_{k+1} = f_k(x_k, u_k, \omega_k)$
  - $k$: time
  - $x_k$: state
  - $u_k$: control decision selected at time $k$
  - $\omega_k$: a random noise
- Cost: $g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, \omega_k)$
- Objective: minimize (maximize) the expected cost.
  - We need to take expectation to account for the noise, $\omega_k$. 
Principle of Optimality

- Let $\pi^* = \{\mu_0^*, \mu_1^*, \ldots, \mu_{N-1}^*\}$ be an optimal policy for the basic problem, and assume that when using $\pi^*$, a given state $x_i$ occurs at time $i$ with positive probability. Consider the sub-problem whereby we are at $x_i$ at time $i$ and wish to minimize the “cost-to-go” from time $i$ to time $N$.

- $E\{g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, \omega_k)\}$

- Then the truncated policy $\{\mu_i^*, \mu_{i+1}^*, \ldots, \mu_{N-1}^*\}$ is optimal for this sub-problem.
Dynamic Programming Algorithm

- For every initial state $x_0$, the optimal cost $J^*(x_k)$ of the basic problem is equal to $J_0(x_0)$, given by the last step of the following algorithm, which proceeds backward in time from period $N - 1$ to period $0$:
  - $J_N(x_N) = g_N(x_N)$
  - $J_k(x_k) = \min_{u_k} E\{g_k(x_k, u_k, \omega_k) + J_{k+1}(f_k(x_k, u_k, \omega_k))\}$
Value function

- Terminal condition:
  \[ G(T, V, x) = V^Y \]

- DP equation:
  \[ G(t, V_t, x_t) = \max_{h_t} E\{G(t + dt, V_{t+dt}, x_{t+dt})\} \]
  \[ G(t, V_t, x_t) = \max_{h_t} E\{G(t, V_t, x_t) + \Delta G\} \]

- By Ito's lemma:
  \[ \Delta G = G_t dt + G_V (dV) + G_x (dx) + \frac{1}{2} G_{VV} (dV)^2 + \frac{1}{2} G_{xx} (dx)^2 + G_{Vx} (dV)(dx) \]
Hamilton-Jacobi-Bellman Equation

- Cancel $G(t, V_t, x_t)$ on both LHS and RHS.
- Divide by time discretization, $\Delta t$.
- Take limit as $\Delta t \to 0$, hence Ito.
- $0 = \max_{h_t} E\{\Delta G\}$
- $\max_{h_t} E \left\{ G_t dt + G_V (dV) + G_x (dx) + \frac{1}{2} G_{VV} (dV)^2 + \frac{1}{2} G_{xx} (dx)^2 + G_{Vx} (dV)(dx) \right\} = 0$
- The optimal portfolio position is $h_t^*$. 
HJB for Our Portfolio Value

- \[
\max_{h_t} E \left\{ G_t dt + G_V (dV) + G_x (dx) + \frac{1}{2} G_{VV} (dV)^2 + \frac{1}{2} G_{xx} (dx)^2 + G_{Vx} (dV)(dx) \right\} = 0
\]

- \[
\max_{h_t} E \left\{ \frac{1}{2} G_{VV} (h_t k(\theta - x_t) dt + h_t \eta d\omega_t)^2 + \frac{1}{2} G_{xx} (dx)^2 + G_{Vx} (h_t k(\theta - x_t) dt + h_t \eta d\omega_t)(dx) \right\} = 0
\]

- \[
\max_{h_t} E \left\{ G_t dt + G_V (h_t k(\theta - x_t) dt + h_t \eta d\omega_t) + G_x (k(\theta - x_t) dt + \eta d\omega_t) + \frac{1}{2} G_{VV} (h_t k(\theta - x_t) dt + h_t \eta d\omega_t)^2 + \frac{1}{2} G_{xx} (k(\theta - x_t) dt + \eta d\omega_t)^2 + G_{Vx} (h_t k(\theta - x_t) dt + h_t \eta d\omega_t) \times (k(\theta - x_t) dt + \eta d\omega_t) \right\} = 0
\]
Taking Expectation

- All $\eta d\omega_t$ disappear because of the expectation operator.
- Only the deterministic $dt$ terms remain.
- Divide LHR and RHS by $dt$.

$$\max_{h_t} \left\{ G_t + G_V (h_t k(\theta - x_t)) + G_x (k(\theta - x_t)) + \frac{1}{2} G_{VV} (h_t \eta)^2 + \frac{1}{2} G_{xx} \eta^2 \right\} + G_{Vx} (h_t \eta^2) = 0$$
Dynamic Programming Solution

- Solve for the cost-to-go function, $G_t$.
- Assume that the optimal policy is $h_t^*$. 
First Order Condition

- Differentiate w.r.t. $h_t$.

$$G_V(k(\theta - x_t)) + h_t^* G_{VV} \eta^2 + G_{Vx} \eta^2 = 0$$

$$h_t^* = -\frac{G_V(k(\theta - x_t)) + G_{Vx} \eta^2}{G_{VV} \eta^2}$$

- In order to determine the optimal position, $h_t^*$, we need to solve for $G$ to get $G_V$, $G_{Vx}$, and $G_{VV}$. 
The Partial Differential Equation (1)

\[ G_t - \]
\[ G_V \left( \frac{G_V(k(\theta - x_t)) + G_{Vx}\eta^2}{G_{VV}\eta^2} \right) k(\theta - x_t) + \]
\[ G_x \left( k(\theta - x_t) \right) + \]
\[ \frac{1}{2} G_{VV}\eta^2 \left( \frac{G_V(k(\theta - x_t)) + G_{Vx}\eta^2}{G_{VV}\eta^2} \right)^2 + \]
\[ \frac{1}{2} G_{xx}\eta^2 - \]
\[ G_{Vx}\eta^2 \frac{G_V(k(\theta - x_t)) + G_{Vx}\eta^2}{G_{VV}\eta^2} = 0 \]
The Partial Differential Equation (2)

\[ G_t = G_V k(\theta - x_t) \frac{G_V (k(\theta - x_t) + G_{Vx} \eta^2)}{G_{VV} \eta^2} + G_x \left( k(\theta - x_t) \right) + \frac{1}{2} \left[ G_V (k(\theta - x_t) + G_{Vx} \eta^2) \right]^2 \frac{1}{G_{VV} \eta^2} + \frac{1}{2} G_{xx} \eta^2 - G_{Vx} \frac{G_V (k(\theta - x_t) + G_{Vx} \eta^2)}{G_{VV}} = 0 \]
Dis-equilibrium

Let $b = k(\theta - x_t)$. Rewrite:

$$G_t - G_V b \frac{G_V b + G_{Vx} \eta^2}{G_{VV} \eta^2} + G_x b + \frac{1}{2} \left[ \frac{G_V b + G_{Vx} \eta^2}{G_{VV} \eta^2} \right]^2 + \frac{1}{2} G_{xx} \eta^2 - G_{Vx} \frac{G_V b + G_{Vx} \eta^2}{G_{VV}} = 0$$

Multiply by $G_{VV} \eta^2$.

$$G_t G_{VV} \eta^2 - G_V b (G_V b + G_{Vx} \eta^2) + G_x b G_{VV} \eta^2 + \frac{1}{2} \left[ \frac{G_V b + G_{Vx} \eta^2}{G_{VV} \eta^2} \right]^2 + \frac{1}{2} G_{xx} G_{VV} \eta^4 - G_{Vx} \eta^2 [G_V b + G_{Vx} \eta^2] = 0$$
Simplification

- Note that

\[-G_V b (G_V b + G_{Vx} \eta^2) + \frac{1}{2} [G_V b + G_{Vx} \eta^2]^2 -
G_{Vx} \eta^2 [G_V b + G_{Vx} \eta^2] = -\frac{1}{2} (G_V b + G_{Vx} \eta^2)^2\]

- The PDE becomes

\[G_t G_{VV} \eta^2 + G_x b G_{VV} \eta^2 + \frac{1}{2} G_{xx} G_{VV} \eta^4 -
\frac{1}{2} (G_V b + G_{Vx} \eta^2)^2 = 0\]
The Partial Differential Equation (3)

\[ G_t G_{VV} \eta^2 + G_x b G_{VV} \eta^2 + \frac{1}{2} G_{xx} G_{VV} \eta^4 - \frac{1}{2} (G_V b + G_{Vx} \eta^2)^2 = 0 \]
Ansatz for G

\[ G(t, V, x) = f(t, x)V^\gamma \]
\[ G(T, V, x) = V^\gamma \]
\[ f(T, x) = 1 \]
\[ G_t = V^\gamma f_t \]
\[ G_V = \gamma V^{\gamma-1} f \]
\[ G_{VV} = \gamma (\gamma - 1)V^{\gamma-2} f \]
\[ G_x = V^\gamma f_x \]
\[ G_{Vx} = \gamma V^{\gamma-1} f_x \]
\[ G_{xx} = V^\gamma f_{xx} \]
Another PDE (1)

\[ V^\gamma f_t \gamma (\gamma - 1) V^{\gamma - 2} f \eta^2 + V^\gamma f_x b \gamma (\gamma - 1) V^{\gamma - 2} f \eta^2 + \]
\[ \frac{1}{2} V^\gamma f_{xx} \gamma (\gamma - 1) V^{\gamma - 2} f \eta^4 - \]
\[ \frac{1}{2} (\gamma V^{\gamma - 1} f b + \gamma V^{\gamma - 1} f_x \eta^2)^2 = 0 \]

Divide by \( \gamma (\gamma - 1) \eta^2 V^{2\gamma - 2} \).

\[ f_t f + f_x b f + \frac{1}{2} f_{xx} f \eta^2 - \frac{\gamma}{2(\gamma - 1)} \left( f \frac{b}{\eta} + f_x \eta \right)^2 = 0 \]
Ansatz for $f$

\[
ff_t + bff_x + \frac{1}{2} \eta^2 ffx - \frac{\gamma}{2(\gamma-1)} \left( \frac{b}{\eta} f + \eta f_x \right)^2 = 0
\]

\[
f(t, x) = g(t) \exp[x\beta(t) + x^2 \alpha(t)] = g \exp(x\beta + x^2 \alpha)
\]

\[
f_t = g_t \exp(x\beta + x^2 \alpha) + g \exp(x\beta + x^2 \alpha) (x\beta_t + x^2 \alpha_t)
\]

\[
f_x = g \exp(x\beta + x^2 \alpha) (\beta + 2x\alpha)
\]

\[
f_{xx} = g \exp(x\beta + x^2 \alpha) (\beta + 2x\alpha)^2 + g \exp(x\beta + x^2 \alpha) (2\alpha)
\]

\[
f_x \quad f = \beta + 2\alpha x
\]
Boundary Conditions

\[ f(T, x) = g(T) \exp[x\beta(T) + x^2\alpha(T)] = 1 \]
\[ g(T) = 1 \]
\[ \alpha(T) = 0 \]
\[ \beta(T) = 0 \]
Yet Another PDE (1)

- \( g \exp(x\beta + x^2\alpha) [g_t \exp(x\beta + x^2\alpha) + g \exp(x\beta + x^2\alpha) (x\beta_t + x^2\alpha_t)] + b g \exp(x\beta + x^2\alpha) g \exp(x\beta + x^2\alpha) (\beta + 2x\alpha) + \frac{1}{2} \eta^2 g \exp(x\beta + x^2\alpha) [g \exp(x\beta + x^2\alpha) (\beta + 2x\alpha)^2 + g \exp(x\beta + x^2\alpha) (2\alpha)] - \)

\[ \frac{\gamma}{2(\gamma-1)} \left( \frac{b}{\eta} g \exp(x\beta + x^2\alpha) + \eta g \exp(x\beta + x^2\alpha) (\beta + 2x\alpha) \right)^2 = 0 \]

- Divide by \( g \exp(x\beta + x^2\alpha) \exp(x\beta + x^2\alpha) \).

- \([g_t + g(x\beta_t + x^2\alpha_t)] + b g(\beta + 2x\alpha) + \frac{1}{2} \eta^2 [g(\beta + 2x\alpha)^2 + g(2\alpha)] - \)

\[ \frac{\gamma}{2(\gamma-1)} g \left( \frac{b}{\eta} + \eta (\beta + 2x\alpha) \right)^2 = 0 \]

- \( g_t + g(x\beta_t + x^2\alpha_t) + b g(\beta + 2x\alpha) + \frac{1}{2} \eta^2 g(\beta + 2x\alpha)^2 + \eta^2 g \alpha - \)

\[ \frac{\gamma}{2(\gamma-1)} g \left( \frac{b}{\eta} + \eta (\beta + 2x\alpha) \right)^2 = 0 \]
Yet Another PDE (2)

- $\lambda = \frac{\gamma}{2(\gamma-1)}$

- $g_t + g(x\beta_t + x^2\alpha_t) + bg(\beta + 2x\alpha) +$
  \[\frac{1}{2}\eta^2 g(\beta + 2x\alpha)^2 + \eta^2 g\alpha - \lambda g \left(\frac{b}{\eta} + \eta(\beta + 2x\alpha)\right)^2 = 0\]
Expansion in $x$

- $g_t + gx\beta_t + gx^2\alpha_t + bg\beta + 2x\alpha bg + \frac{1}{2} \eta^2 g(\beta^2 + 4x^2\alpha^2 + 4x\alpha\beta) + \eta^2 g\alpha - \lambda g \left( \frac{b^2}{\eta^2} + \eta^2 \beta^2 + 4\eta^2x^2\alpha^2 + 2b\beta + 4bx\alpha + 4\eta^2x\alpha\beta \right) = 0$

- $g_t + gx\beta_t + gx^2\alpha_t + k(\theta - x)g\beta + 2x\alpha k(\theta - x)g + \frac{1}{2} \eta^2 g(\beta^2 + 4x^2\alpha^2 + 4x\alpha\beta) + \eta^2 g\alpha - \lambda g \left( \frac{k^2(\theta-x)^2}{\eta^2} + \eta^2 \beta^2 + 4\eta^2x^2\alpha^2 + 2k(\theta - x)\beta + 4k(\theta - x)x\alpha + 4\eta^2x\alpha\beta \right) = 0$

- $g_t + gx\beta_t + gx^2\alpha_t + k g\beta\theta - kg\beta x + 2x\alpha k g\theta - 2akg x^2 + \frac{1}{2} \eta^2 g\beta^2 + 2\eta^2 gx^2\alpha^2 + 2\eta^2 gx\alpha\beta + \eta^2 g\alpha - \frac{\lambda g}{\eta^2} k^2 \theta^2 + 2 \frac{\lambda g}{\eta^2} k^2 x \theta - \frac{\lambda g}{\eta^2} k^2 x^2 - \lambda g \eta^2 \beta^2 - 4\lambda g \eta^2 x^2 \alpha^2 - 2\lambda g k \beta \theta + 2\lambda g k \beta x - 4\lambda g k \theta x \alpha + 4\lambda g k x \alpha^2 - 4\lambda g \eta^2 x \alpha \beta = 0$
Grouping in $x$

\[
\left( g_t + kg\beta \theta + \frac{1}{2} \eta^2 g \beta^2 + \eta^2 g\alpha - \frac{\lambda g}{\eta^2} k^2 \theta^2 - \lambda g \eta^2 \beta^2 - 2\lambda g k \beta \theta \right) + \\
\left( g\beta_t - kg\beta + 2\alpha k g \theta + 2\eta^2 g\alpha \beta + 2 \frac{\lambda g}{\eta^2} k^2 \theta + 2\lambda g k \beta - 4\lambda g k \theta \alpha - 4\lambda g \eta^2 \alpha \beta \right) x + \\
\left( g\alpha_t - 2\alpha k g + 2\eta^2 g\alpha^2 - \frac{\lambda g}{\eta^2} k^2 - 4\lambda g \eta^2 \alpha^2 + 4\lambda g k \alpha \right) x^2 = 0
\]
The Three PDE’s (1)

\[ g\alpha_t - 2\alpha k g + 2\eta^2 g\alpha^2 - \frac{\lambda g}{\eta^2} k^2 - 4\lambda g\eta^2 \alpha^2 + 4\lambda g k \alpha = 0 \]

\[ g\beta_t - k g \beta + 2\alpha k g \theta + 2\eta^2 g\alpha \beta + 2\frac{\lambda g}{\eta^2} k^2 \theta + 2\lambda g k \beta - 4\lambda g k \theta \alpha - 4\lambda g \eta^2 \alpha \beta = 0 \]

\[ g_t + k g \beta \theta + \frac{1}{2} \eta^2 g \beta^2 + \eta^2 g \alpha - \frac{\lambda g}{\eta^2} k^2 \theta^2 - \lambda g \eta^2 \beta^2 - 2\lambda g k \beta \theta = 0 \]
PDE in $\alpha$

- $\alpha_t + (2\eta^2 - 4\lambda\eta^2)\alpha^2 + (4\lambda k - 2k)\alpha - \frac{\lambda}{\eta^2} k^2 = 0$

- $\alpha_t = \frac{\lambda}{\eta^2} k^2 + 2k(1 - 2\lambda)\alpha + 2\eta^2(2\lambda - 1)\alpha^2$
PDE in $\beta$, $\alpha$

\[ \begin{align*}
\beta_t - k\beta + 2\eta^2\alpha\beta + 2\lambda k\beta - 4\lambda\eta^2\alpha\beta - 4\lambda k\theta\alpha + \\
2\frac{\lambda}{\eta^2} k^2\theta + 2\alpha k\theta &= 0
\end{align*} \]

\[ \begin{align*}
\beta_t = \\
(k - 2\eta^2\alpha - 2\lambda k + 4\lambda\eta^2\alpha)\beta + \\
\left(4\lambda k\theta\alpha - 2\frac{\lambda}{\eta^2} k^2\theta - 2\alpha k\theta\right)
\end{align*} \]
PDE in $\beta, \alpha, g$

- $g_t + kg\beta \theta + \frac{1}{2} \eta^2 g \beta^2 + \eta^2 g \alpha - \frac{\lambda g}{\eta^2} k^2 \theta^2 - \lambda g \eta^2 \beta^2 - 2\lambda g k \beta \theta = 0$

- $g_t = -kg\beta \theta - \frac{1}{2} \eta^2 g \beta^2 - \eta^2 g \alpha + \frac{\lambda g}{\eta^2} k^2 \theta^2 + \lambda g \eta^2 \beta^2 + 2\lambda g k \beta \theta$

- $g_t = g \left( -k\beta \theta - \frac{1}{2} \eta^2 \beta^2 - \eta^2 \alpha + \frac{\lambda}{\eta^2} k^2 \theta^2 + \lambda \eta^2 \beta^2 + 2\lambda k \beta \theta \right)$
Riccati Equation

- A Riccati equation is any ordinary differential equation that is quadratic in the unknown function.

\[ \alpha_t = \frac{\lambda}{\eta^2} k^2 + 2k(1 - 2\lambda)\alpha + 2\eta^2(2\lambda - 1)\alpha^2 \]

\[ \alpha_t = A_0 + A_1 \alpha + A_2 \alpha^2 \]
Solving a Riccati Equation by Integration

- Suppose a particular solution, $\alpha_1$, can be found.
- $\alpha = \alpha_1 + \frac{1}{z}$ is the general solution, subject to some boundary condition.
Either $\alpha_1$ or $\alpha_2$ is a particular solution to the ODE. This can be verified by mere substitution.

$$\alpha_{1,2} = \frac{-A_1 \pm \sqrt{A_1^2 - 4A_2A_0}}{2A_2}$$
**z Substitution**

- Suppose $\alpha = \alpha_1 + \frac{1}{z}$.

\[
\left( \frac{1}{z} \right) = A_0 + A_1 \left( \alpha_1 + \frac{1}{z} \right) + A_2 \left( \alpha_1 + \frac{1}{z} \right)^2
\]

\[
= A_0 + \left( A_1 \alpha_1 + A_1 \frac{1}{z} \right) + \left( A_2 \alpha_1^2 + A_2 \frac{1}{z^2} + 2A_2 \frac{\alpha_1}{z} \right)
\]

\[
= A_0 + \left( A_1 \alpha_1 + A_1 \frac{1}{z} \right) + \left( A_2 \alpha_1^2 + A_2 \frac{1}{z^2} + 2A_2 \frac{\alpha_1}{z} \right)
\]

\[
= \left( A_0 + A_1 \alpha_1 + A_2 \alpha_1 \frac{1}{z} \right) + \left( \frac{A_1 + 2\alpha_1 A_2}{z} \right) + \frac{A_2}{z^2}
\]

goes to 0 by the definition of $\alpha_1$
Solving $z$

- \[ \left( \frac{1}{z} \right) = \left( \frac{A_1 + 2\alpha_1 A_2}{z} \right) + \frac{A_2}{z^2} \]
- \[ -\frac{1}{z^2} \dot{z} = \left( \frac{A_1 + 2\alpha_1 A_2}{z} \right) + \frac{A_2}{z^2} \]

1\textsuperscript{st} order linear ODE

- \[ \dot{z} + (A_1 + 2\alpha_1 A_2)z = -A_2 \]
- \[ z(t) = \frac{-A_2}{A_1 + 2\alpha_1 A_2} + C \exp(- (A_1 + 2\alpha_1 A_2)t) \]
Solving for $\alpha$

\[ \alpha = \alpha_1 + \frac{1}{-A_2} + \frac{1}{A_1 + 2\alpha_1 A_2} + C \exp(-(A_1 + 2\alpha_1 A_2)t) \]

boundary condition:

\[ \alpha(T) = 0 \]

\[ \alpha_1 + \frac{1}{-A_2} + \frac{1}{A_1 + 2\alpha_1 A_2} + C \exp(-(A_1 + 2\alpha_1 A_2)T) = 0 \]

\[ C \exp(-(A_1 + 2\alpha_1 A_2)T) = -\frac{1}{\alpha_1} + \frac{A_2}{A_1 + 2\alpha_1 A_2} \]

\[ C = \exp((A_1 + 2\alpha_1 A_2)T) \left[ \frac{A_2}{A_1 + 2\alpha_1 A_2} - \frac{1}{\alpha_1} \right] \]

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α Solution (1)

\[ \alpha = \alpha_1 + \frac{1}{-\frac{A_2}{A_1+2\alpha_1 A_2} + c \exp(-(A_1+2\alpha_1 A_2)t)} \]

= 
\[ \alpha_1 + \frac{1}{-\frac{A_2}{A_1+2\alpha_1 A_2} + \exp((A_1+2\alpha_1 A_2)T)\left[\frac{A_2}{A_1+2\alpha_1 A_2} - \frac{1}{\alpha_1}\right] \exp(-(A_1+2\alpha_1 A_2)t)} \]

= \alpha_1 + \frac{1}{-\frac{A_2}{A_1+2\alpha_1 A_2} + \exp((A_1+2\alpha_1 A_2)(T-t))\left[\frac{A_2}{A_1+2\alpha_1 A_2} - \frac{1}{\alpha_1}\right]} \]

= \alpha_1 + \frac{\alpha_1 (A_1+2\alpha_1 A_2)}{-\alpha_1 A_2 + \exp((A_1+2\alpha_1 A_2)(T-t))(\alpha_1 A_2 - A_1 - 2\alpha_1 A_2)}
\( \alpha \) Solution (2)

\[
\alpha = \alpha_1 + \frac{\alpha_1 (A_1 + 2 \alpha_1 A_2)}{-\alpha_1 A_2 + \exp((A_1 + 2 \alpha_1 A_2)(T-t))(-A_1 - \alpha_1 A_2)}
\]

\[
= \alpha_1 \left[ 1 - \frac{A_1 + 2 \alpha_1 A_2}{A_2 + \exp((A_1 + 2 \alpha_1 A_2)(T-t))\left(\frac{A_1}{\alpha_1} + A_2\right)} \right]
\]

\[
= \alpha_1 \left[ 1 - \frac{\frac{A_1}{A_2} + 2 \alpha_1}{1 + \left(\frac{A_1}{\alpha_1 A_2} + 1\right)\exp((A_1 + 2 \alpha_1 A_2)(T-t))} \right]
\]
\[ \alpha(t) = \frac{k}{2\eta^2} \left[ \left( 1 - \sqrt{1 - \gamma} \right) + \frac{2\sqrt{1 - \gamma}}{1 + \left( 1 - \frac{2}{1 - \sqrt{1 - \gamma}} \right) \exp \left( \frac{2k}{\sqrt{1 - \gamma}} (T - t) \right)} \right] \]
Solving $\beta$

- $\beta_t = (k - 2\eta^2\alpha - 2\lambda k + 4\lambda\eta^2\alpha)\beta + \left(4\lambda k\theta\alpha - 2\frac{\lambda}{\eta^2} k^2\theta - 2\alpha k\theta\right)$
- Let $\tau = T - t$
- $\hat{\beta}(\tau) = \beta(T - t)$
- $\hat{\beta}_\tau(\tau) = -\beta_t(\tau)$
- $-\hat{\beta}_\tau(\tau) = \beta_t(\tau) = \left(k - 2\eta^2\alpha(\tau) - 2\lambda k + 4\lambda\eta^2\alpha(\tau)\right)\beta(\tau) + \left(4\lambda k\theta\alpha(\tau) - 2\frac{\lambda}{\eta^2} k^2\theta - 2\alpha(\tau)k\theta\right)$
- $\hat{\beta}_\tau(\tau) =$
  $\left((-k + 2\eta^2\hat{\alpha} + 2\lambda k - 4\lambda\eta^2\hat{\alpha})\hat{\beta} + \left(-4\lambda k\theta\hat{\alpha} + 2\frac{\lambda}{\eta^2} k^2\theta + 2\hat{\alpha} k\theta\right)\right)$
- $\hat{\beta}_\tau(\tau) =$
  $\left((2\lambda - 1)k + 2\eta^2\hat{\alpha}(1 - 2\lambda)\right)\hat{\beta} + \left(2\hat{\alpha} k\theta(1 - 2\lambda) + 2\frac{\lambda}{\eta^2} k^2\theta\right)$
First Order Non-Constant Coefficients

\[ \hat{\beta}_\tau = B_1 \hat{\beta} + B_2 \]

\[ B_1(\tau) = (2\lambda - 1)k + 2\eta^2 \hat{\alpha}(1 - 2\lambda) \]

\[ B_2(\tau) = 2\hat{\alpha}k\theta(1 - 2\lambda) + 2 \frac{\lambda}{\eta^2} k^2 \theta \]
Integrating Factor (1)

- $\hat{\beta}_\tau - B_1 \hat{\beta} = B_2$
- We try to find an integrating factor $\mu = \mu(\tau)$ s.t.
  - $\frac{d}{d\tau} (\mu \hat{\beta}) = \mu \frac{d\hat{\beta}}{d\tau} + \hat{\beta} \frac{d\mu}{d\tau} = \mu B_2$
- Divide LHS and RHS by $\mu \hat{\beta}$.
  - $\frac{1}{\hat{\beta}} \frac{d\hat{\beta}}{d\tau} + \frac{1}{\mu} \frac{d\mu}{d\tau} = \frac{B_2}{\hat{\beta}}$
- By comparison,
  - $-B_1 = \frac{1}{\mu} \frac{d\mu}{d\tau}$
Integrating Factor (2)

- \[ \int -B_1 d\tau = \int \frac{d\mu}{\mu} = \log \mu + C \]
- \[ \mu = \exp(\int -B_1 d\tau) \]
- \[ \mu \hat{\beta} = \int \mu B_2 d\tau + C \]
- \[ \hat{\beta} = \frac{\int \mu B_2 d\tau + C}{\mu} \]
- \[ \hat{\beta} = \frac{\int \exp(\int -B_1 du) B_2 d\tau + C}{\exp(\int -B_1 d\tau)} \]
\hat{\beta} \text{ Solution}

\[ \hat{\beta} = \frac{\int_0^\tau \exp(\int_0^\tau -B_1(u)du)B_2(s)ds}{\exp(\int_0^\tau -B_1(u)du)} + C \]

\[ \hat{\beta}(\tau) = \exp(\int_0^\tau B_1(u)du) \int_0^\tau \exp(\int_0^s -B_1(u)du) B_2(s) ds + C \]
\[ B_1, B_2 \]

\[ \int_0^\tau B_1(s)ds = \int_0^\tau [(2\lambda - 1)k + 2\eta^2(1 - 2\lambda)\alpha(s)]ds \]

\[ I_2 = \int_0^\tau \exp\left(\int_0^s -B_1(u)du\right) B_2(s) ds \]
\[ \beta(t) = \frac{k \theta}{\eta^2} (1 + \sqrt{1 - \gamma}) \frac{\exp\left(\frac{2k}{\sqrt{1-\gamma}}(T-t)\right) - 1}{1 + \left[1 - \frac{2}{1 - \sqrt{1-\gamma}}\right] \exp\left(\frac{2k}{\sqrt{1-\gamma}}(T-t)\right)} \]
Solving $g$

$g_t = g \left( -k\beta\theta - \frac{1}{2}\eta^2\beta^2 - \eta^2\alpha + \frac{\lambda}{\eta^2} k^2\theta^2 + \lambda\eta^2\beta^2 + 2\lambda k\beta\theta \right)$

With $\alpha$ and $\beta$ solved, we are now ready to solve $g_t$.

$\frac{g_t}{g} = G(t)$

$\frac{d}{ds} \log g_t = \frac{g_t}{g} = G$

$\log g_t = \int Gds + C$

$g_t = C \exp(\int Gdt) = \exp \left(- \int_0^T Gds \right) \exp \left( \int_0^t Gds \right)$

$g_t = \exp \left(- \int_t^T Gds \right)$
Computing the Optimal Position

\[ h(t)^* = - \frac{G_V(k(\theta-x_t)) + G_V x \eta^2}{G_{VV} \eta^2} \]

\[ = - \frac{\gamma V^\gamma f(k(\theta-x_t)) + \gamma V^\gamma f_x \eta^2}{\gamma (\gamma-1) V^\gamma - 2 f \eta^2} \]

\[ = - \frac{V f(k(\theta-x_t)) + V f_x \eta^2}{(\gamma-1) f \eta^2} \]

\[ = - \frac{V}{(\gamma-1) \eta^2} \left[ f(k(\theta-x_t)) + f_x \eta^2 \right] \]

\[ = \frac{V}{(1-\gamma) \eta^2} \left[ k(\theta - x_t) + \eta^2 (\beta + 2\alpha x) \right] \]

\[ = \frac{V}{(1-\gamma)} \left[ - \frac{k}{\eta^2} (x_t - \theta) + 2\alpha x + \beta \right] \]
The Optimal Position

\[ h(t)^* = \frac{V_t}{(1-\gamma)} \left[ -\frac{k}{\eta^2} (x_t - \theta) + 2\alpha(t)x_t + \beta(t) \right] \]

\[ h(t)^* \sim -\frac{k}{\eta^2} (x_t - \theta) \]
P&L for Simulated Data

- The portfolio increases from $1000 to $4625 in one year.
Parameter Estimation

- Can be done using Maximum Likelihood.
- Evaluation of parameter sensitivity can be done by Monte Carlo simulation.
- In real trading, it is better to be conservative about the parameters.
  - Better underestimate the mean-reverting speed
  - Better overestimate the noise
- To account for parameter regime changes, we can use:
  - a hidden Markov chain model
  - moving calibration window